Journal of Sound and Vibration 328 (2009) 618–619

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Discussion

Comments on "A critical evaluation of closure methods via two simple dynamic systems"

Roman V. Bobryk *

Institute of Mathematics, Jan Kochanowski University, 25406 Kielce, Poland

ARTICLE INFO

Article history:
Accepted 26 August 2009
The peer review of this article was organized by the Guest Editor
Available online 19 September 2009

Study of randomly excited dynamical systems such as random vibrations often leads to infinite hierarchies for relevant probabilistic characteristics (e.g., moments). In practice, to solve the arising hierarchy one usually transforms it into a finite system of equations by applying a so-called closure procedure. Such procedures are not always justified and sometimes generate physically inconsistent solutions to the resulting finite systems.

In an interesting recent article [1], Grigoriu considered some closure procedures for infinite moment hierarchy of the overdamped nonlinear oscillator

$$\mathrm{d}X(t) = (\alpha X(t) + \beta X^3(t))\,\mathrm{d}t + \sigma\,\mathrm{d}w(t), \quad t \geq 0, \tag{1}$$

where α and σ are real numbers, β is negative, and $w(t)$ is a standard Wiener process. The main finding of [1] is somewhat unexpected: "the performance of closure methods is determined by the structure of the moment equations rather than the way in which the infinite hierarchy of moment equations are closed, that is the particular closure method used for solution".

In this letter we present a mathematical explanation to the numerical results in [1], namely, we give an elementary proof of the non-uniqueness of solution to the moment hierarchy associated with (1). More precisely, we show that this hierarchy has infinitely many solutions with at least one being physically inconsistent.

It is well known that the solution $X(t)$ to Eq. (1) has the following stationary density:

$$f(x) = \exp\left\{\sigma^{-2}\left(\alpha x^2 + \frac{\beta}{2}x^4\right)\right\} \int_{\infty}^{\infty} \exp\left\{\sigma^{-2}\left(\alpha x^2 + \frac{\beta}{2}x^4\right)\right\}\mathrm{d}x, \tag{2}$$

so that all odd moments of $X(t)$ vanish. The corresponding even moments,

$$\mu_{2k}(\alpha,\beta,\sigma) = \int_{\infty}^{\infty} x^{2k} f(x)\,\mathrm{d}x, \quad k \in \mathbb{N}, \tag{3}$$

satisfy the following hierarchy [1], which can be easily obtained integrating by parts in (3):

$$ay_2(\alpha,\beta,\sigma) - y_4(\alpha,\beta,\sigma) + b = 0,$$

* DOI of original articles:10.1016/j.jsv.2009.08.034,10.1016/j.jsv.2008.02.049
Tel.: +48 4 13496264; fax: +48 4 13496254.
E-mail address: bobryk@ujk.kielce.pl

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.08.027

$$ay_{2k}(\alpha,\beta,\sigma) - y_{2k+2}(\alpha,\beta,\sigma) + (2k-1)b\,y_{2k-2}(\alpha,\beta,\sigma) = 0, \quad k \geq 2, \tag{4}$$

where we write $a = -\alpha/\beta$ and $b = -\sigma^2/(2\beta)$.

Is the moment sequence (3) a unique solution to this hierarchy? The answer is negative. Indeed, one can easily verify that together with the sequence $\mu_{2k}(\alpha,\beta,\sigma)$, $k \in \mathbb{N}$, the sequence

$$(-1)^k \mu_{2k}(-\alpha,\beta,\sigma), \quad k \in \mathbb{N}, \tag{5}$$

also satisfies the hierarchy (4). The sequence (5) gives a physically inconsistent solution because it has both positive and negative elements. Moreover, one can easily check that if sequences $y_{2k}^{(1)}$ and $y_{2k}^{(2)}$ satisfy this hierarchy, then the sequence $(y_{2k}^{(1)} + y_{2k}^{(2)})/2$ also solves (4). If $y_{2k}^{(1)} = \mu_{2k}(\alpha,\beta,\sigma)$ and $y_{2k}^{(2)} = (-1)^k \mu_{2k}(-\alpha,\beta,\sigma)$, then $(y_{2k}^{(1)} + y_{2k}^{(2)})/2$ gives the third solution. Continuation of this procedure leads to infinitely many solutions of the hierarchy.

To solve the infinite hierarchy (4), Grigoriu [1] used a closure procedure based on the relation

$$y_{2n+2}(\alpha,\beta,\sigma) = \gamma\, y_{2n}(\alpha,\beta,\sigma), \quad \gamma > 0, \tag{6}$$

introduced in the n th equation of the hierarchy. We investigated numerically the resulting finite system using *Mathematica* using the same values of parameters α, β and σ as in [1].

For $\alpha = -1$, $\beta = -1$ and $\sigma = 1$ (single well potential case) the numerical value of the second moment μ_2 is about 0.289602. The same value can be obtained by solving the finite linear system closed at $n = 75$ and $\gamma = 5$, with exactly the same value obtained for the closure index $n = 150$.

On the other hand, for $\alpha = 1$, $\beta = -1$ and $\sigma = 1$ (double well potential case) the numerical value of the second moment μ_2 is about 0.893465. However, solving the finite linear system closed at $n = 70$ and $\gamma = 5$ we obtain a physically inconsistent value -0.289602 (and exactly the same value is obtained for the closure index $n = 140$).

Our numerical results suggest that the closure procedure (6) from [1] gives physically consistent solution (3) in the single well potential case and physically inconsistent solution (5) in the double well potential case.

Having conducted similar numerical experiments using the cumulant-neglect closure method and the high-level Gaussian closure procedure, we obtained analogous results; namely, in the single well potential case a physically consistent solution was obtained, whereas the double well potential case resulted in a physically inconsistent one.

We hope that this comment clarifies some obscurities in application of closure procedures to more complicated nonlinear stochastic systems as in [2,3]. However, the question of finding a closure procedure leading to physically consistent solutions for systems with double well potentials remains open.

References

[1] M. Grigoriu, A critical evaluation of closure methods via two simple dynamic system, *Journal of Sound and Vibration* 317 (2008) 190–198.

[2] S.F. Wojtkiewicz, B.F. Spencer, L.A. Bergman, On the cumulant-neglect closure method in stochastic dynamics, *International Journal of Non-Linear Mechanics* 31 (1996) 657–684.

[3] G.I. Schuëller, A state-of-the-art report on computational stochastic mechanics, *Probabilistic Engineering Mechanics* 12 (1997) 197–321.